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ON DECOMPOSITION OF LATTICE Ideals of A LATTICE-ORDERED SEMIGROUP

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Our purpose of the present note is to obtain a unique decomposition theorem of lattice ideals of l-semigroups treated in [2]. The decomposition theorem is a generalization of the unique factorization of elements in the arithmetical l-groups [7]. Applying our theorem to submodules over a maximal bounded order of a ring, we obtain a decomposition of the modules [5].

1. PRELIMINARIES. Let $L = (L, \cdot, \leq)$ be a (conditionally) complete l-semigroup with multiplicative unity e. We assume the following two conditions:

(1) L has a map $a \mapsto a^{-1}$ into itself with two properties (i) $aa^{-1}a \leq a$ and (ii) $axa \leq a$ implies $a \leq a^{-1}$.

(2) e is maximally integral: $c^2 \leq c$ and $e \leq c$ imply $c = e$.

For any a of L we define $a^* = (a^{-1})^{-1}$, and define $a^* \circ b^* = (a^* b^*)^* = (ab)^*$ [2]. Then the set $L^* = \{a^*;\ a \in L\}$ is a complete l-group under $\circ$ and $\leq$ [3]. Hence the group $(L^*, \circ)$ is commutative by the well known theorem of l-groups. If we classify L by the quasi-equal relation $a \sim b$ defined by $a^{-1} = b^{-1}$, then the set $L/\sim$ of all cosets forms an l-group canonically and it is isomorphic to $(L^*, \circ, \leq)$. We now put the ascending chain condition in the sense of quasi-equality for integral elements of L. Then we can prove that $p^* = p$ for any prime p which is not quasi-equal to e [2]. In the following $\mathbb{P}$ will denote the set of all primes not quasi-equal to e. Then any element a of L is factored into a finite number of primes:

$$a \sim \prod_{p \in \mathbb{P}} p^{\nu(p,a)}$$

where $\nu(p,a)$ is the p-exponent of a. We have then (1°) $\nu(p,a) = 0$ for all but finite many $p \in \mathbb{P}$, (2°) $a \sim b$ if and only if $\nu(p,a) = \nu(p,b)$ for all $p \in \mathbb{P}$, (3°) $\nu(p,a) = \nu(p,a^*)$, (4°) $\nu(p,ab) = \nu(p,a) + \nu(p,b)$, (5°) $\nu(p,a \cup b) = \min\{\nu(p,a), \nu(p,b)\}$, (6°) $a \leq b^*$ (i.e. $a^* \leq b^*$) if and only if $\nu(p,a) \geq \nu(p,b)$ for all $p \in \mathbb{P}$.

A lattice ideal (abbr. l-ideal) J is called closed if $a \in J$ implies $a^* \in J$. Let A be any non-empty subset of L, and let A' be the join semi-lattice generated by A. Then the set-theoretical union of all principal closed l-ideals $J(a^*)$'s generated by $a \in A'$ is the closed l-ideal generated by A. Let P be any subset of $\mathbb{P}$. If P is non-void, the closed l-ideal generated by $\{p_1^{-1} \cdots p_n^{-1}\ ;\ p_i \in P\}$ is called a P-component of the cone I and denoted by I_P. If P is void, I_P means I itself. A P-component J_P of the closed l-ideal J will be defined to be the closed l-ideal generated by $J \cdot I_P = \{xy;\ x \in J,\ y \in I_P\}$. For convenience the closed l-ideal generated by the l-ideal J will be denoted by J^*. For two l-ideals J_1 and J_2 we define quasi-equal relation by $J_1 \sim J_2 \Longleftrightarrow J_1^* = J_2^*$. $J_1 \circ J_2$ means the closed l-ideal generated by $\{xy;\ x \in J_1,\ y \in J_2\}$ for any two l-ideals J_1 and J_2. Then the set of all closed l-ideals $\mathcal{J} = (\mathcal{J}, \circ, \subseteq)$ forms a complete l-semigroup which contains the cl-semigroup $(L^*, \circ, \leq)$ isomorphically. It can be seen that $(\mathcal{J}, \circ)$ is a commutative semigroup.

The set-theoretical union $Z_{-\infty}$ of the rational integers Z and the symbol $-\infty$ is a totally ordered additive semigroup. For any l-ideal J of L we define

$$\nu(p,J) = \inf\{\nu(p,a);\ a \in J\}.$$

Fixing J and moving p over $\mathbb{P}$, $\nu(p,J)$ is considered as a map from $\mathbb{P}$ into $Z_{-\infty}$. The map is written by ν_J, that is $\nu_J(p) = \nu(p,J)$.

Let now σ be a map from $\mathbb{P}$ into $Z_{-\infty}$ such that $\sigma(p) \leq 0$ for almost all $p \in \mathbb{P}$, and let S be the set of all such maps. Then the set G of all vectors $[\sigma(p)]$ forms a complete l-semigroup under the usual addition and the order $\preceq$ defined by $[\sigma(p)] \preceq [\sigma'(p)] \Longleftrightarrow \sigma(p) \geq \sigma'(p)$ for all $p \in \mathbb{P}$. In symbol: $G = (G, +, \preceq)$.

2. LEMMAS AND MAIN RESULTS.

LEMMA 1. For each $\sigma \in S$, the set $K[\sigma]$ of all $x \in L$ such that $\nu(p,x) \geq \sigma(p)$ for all $p \in \mathbb{P}$ forms a closed l-ideal of L.

Proof. This is immediate by (2°), (5°) and (6°) in Section 1.

LEMMA 2. For each closed l-ideal J we have $K[\nu_J] = J$.

Proof. Similarly obtained as the proof of Lemma 3 in [7].

LEMMA 3. For each $\sigma \in S$ we have $\nu_{K[\sigma]} = \sigma$.

Proof. Similarly obtained as the proof of Lemma 4 in [7].

By using LEMMAS 2 and 3 we obtain the following

THEOREM 1. The map $f\colon J \mapsto f(J) = [\nu_J(p)]$ gives an l-semigroup isomorphism from $(\mathcal{J}, \circ, \subseteq)$ onto $(G, +, \precsim)$.

Let $P_+(J)$, $P_0(J)$, $P_-(J)$ and $P_{-\infty}(J)$ be the sets of primes p in $\mathbb{P}$ such that $\nu_J(p)$ is positive, zero, negative and $-\infty$, respectively.

LEMMA 4. Let J be a closed l-ideal such that both $P_+(J)$ and $P_-(J)$ are void. If $P_0(J)$ is contained in the set-theoretical union of $P_0(J(a))$ and $P_+(J(a))$, then a is contained in J and conversely.

By using Corollary to Theorem 2.3 in [2] we get the following

LEMMA 5. Let J be a closed l-ideal. If J is multiplicatively closed, the vector f(J) has no integral coordinate except zero, and vice versa.

LEMMA 6. Let J be a closed l-ideal containing the cone I. If J is closed under multiplication, J is the $P_{-\infty}(J)$-component of I.

THEOREM 2. Any l-ideal J of L is decomposed as follows:

$$(*) \qquad J \sim \prod_{p \in P_+} J(p^{\nu_p}) \cdot \left(\bigvee J\left(\bigcup_{p \in P_-}' p^{\nu_p}\right)\right) \cdot I_P.$$

where $\nu_p = \nu_J(p)$, $P_+ = P_+(J^*)$, $P_- = P_-(J^*)$, $\bigcup'$ denotes a finite join and $\bigvee$ denotes the set-theoretical union of all $J(\bigcup' p^{\nu_p})$. Conversely, let A, B, C be any three subsets of $\mathbb{P}$ such that they are disjoit and one of them is finite, e. g. so is A, and let α_q and $-\beta_q$ be positive and negative integers respectively such that α_q corresponds $q \in A$ and $-\beta_q$ corresponds to $q \in B$. Then

$$(**) \qquad \prod_{q \in A} J(q^{\alpha_q}) \cdot \left(\bigvee J\left(\bigcup_{q \in B}' q^{-\beta_q}\right)\right) \cdot I_C$$

is an l-ideal of L. Moreover if J of (*) is quasi-equal to (**), then $P_+ = A$, $P_- = B$, $P_{-\infty} = C$, $\nu_p = \alpha_q$ $(p \in P_+)$, $\nu_p = -\beta_q$ $(p \in P_-)$ by suitable enumeration of p; that is, the decomposition (*) is unique within quasi-equality.

Proof. Let J be any l-ideal of L. Firstly we suppose that J is closed. f(J) is represented as $f(J) = u_+(J) + u_-(J) + u_{-\infty}(J)$, where $u_+(J)$, $u_-(J)$, $u_{-\infty}(J)$ are the vectors whose p-coordinates are $\nu_J(p)$ if p is positive-, negative-, $-\infty$-spots (zero otherwise), respectively. It is clear that $f^{-1}(u_+(J)) = \prod^{\circ}_{p \in P_+} J(p^{\nu_p})$. Take any element a of $f^{-1}(u_-(J))$, and let $a^* = p_1^{\lambda_1} \circ \cdots \circ p_n^{\lambda_n}$, $p_i \in \mathbb{P}$. If $\lambda_i \geqq 0$ for all i, then a* is integral, hence so is the element a. Therefore a is contained in $\bigvee J(\mathring{\cup}' p^{\nu_p})$. If $\lambda_1 < 0, \ldots, \lambda_r < 0$, $\lambda_{r+1} > 0, \ldots, \lambda_n > 0$ for r with $0 < r \leq n$, then we obtain $a \leq a^* \leq p_1^{\lambda_1} \circ \cdots \circ p_r^{\lambda_r} \leq (p_1^{\nu_{p_1}} \cup \cdots \cup p_r^{\nu_{p_r}})^* = p_1^{\nu_{p_1}} \mathring{\cup} \cdots \mathring{\cup} p_r^{\nu_{p_r}}$. This implies $a \in J(p_1^{\nu_{p_1}} \mathring{\cup} \cdots \mathring{\cup} p_r^{\nu_{p_r}})$. Hence $f^{-1}(u_-(J)) \subseteq \bigvee J(\mathring{\cup}' p_i^{\nu_{p_i}})$. The converse inclusion is easy to see. Next, by using LEMMAS 5 and 6 we obtain $f^{-1}(u_{-\infty}(J)) = I_{P_{-\infty}(J)}$. The last part of the theorem can be proved easily.

3. APPLICATION.

1. Let R be a noncommutative ring with a bounded maximal order $\mathfrak{O}$, and let $\mathfrak{L}$ be all the non-zero fractional two-sided $\mathfrak{O}$-ideals (abbr. ideals) in R [4]. $\mathfrak{L}$ is then a conditionally complete l-semigroup under module-product and set-inclusion. We assume throughout this paragraph that the ascending cain condition in the sense of

quasi-equality holds for integral ideals [1]. The term submodule means a two-sided $\mathfrak{o}$-submodule of R which contains a regular element of R. A submodule M of R is said to be closed if $\mathfrak{a} \subseteq M$ implies $\mathfrak{a}^* = (\mathfrak{a}^{-1})^{-1} \subseteq M$, where $\mathfrak{a}$ is an ideal and $\mathfrak{a}^{-1}$ is the inverse of $\mathfrak{a}$. The set-theoretical union M* of $\mathfrak{a}^*$ for the ideals $\mathfrak{a}$ contained in M is the closed submodule generated by M. Two submodules M_1 and M_2 are said to be quasi-equal iff $M_1{}^* = M_2{}^*$. In symbol: $M_1 \sim M_2$. If we define $M_1 \cdot M_2$ of any two closed submodules M_1 and M_2 to be the set-theoretical union of all ideals $(\sum_{i=1}^{n} \mathfrak{a}_i \mathfrak{b}_i)^*$ where $\mathfrak{a}_i \subseteq M_1$, $\mathfrak{b}_i \subseteq M_2$, then the set $\mathfrak{M}^* = (\mathfrak{M}^*, \cdot, \subseteq)$ of all closed submodules of R forms a commutative cl-semigroup. If we classify the cl-semigroup $\mathfrak{M}$ consisting of all submodules of R by the quasi-equal relation $\sim$, then $\mathfrak{M}/\sim$, the set of all cosets $[M_1]$, $[M_2]$, ..., is a commutative cl-semigroup which is isomorphic to $(\mathfrak{M}^*, \cdot, \leq)$, where the product of two cosets is the coset containing $(M_1 \cdot M_2)^*$ and the order $\leq$ is defined by $[M_1] \leq [M_2] \Longleftrightarrow M_1{}^* \subseteq M_2{}^*$. Let J be any closed l-ideal of $\mathcal{L}$. Then the set-theoretical union M(J) of all ideals in J is a closed submodule of R. Conversely the correction J(M) of all ideals in the closed submodule M is an l-ideal of $\mathcal{L}$. Then we have $J \mapsto M(J) \mapsto J(M(J)) = J$ and $M \mapsto J(M) \mapsto M(J(M)) = M$. Let $(\mathcal{L}^*, \circ, \subseteq)$ be the cl-semigroup consisting of all closed l-ideals in $\mathcal{L}$, where "$\circ$" is defined as in the former section. Then the map $M \mapsto J(M)$ gives an l-semigroup isomorphism from $(\mathfrak{M}^*, \circ, \leq)$ onto $(\mathcal{L}^*, \circ, \subseteq)$. Under that isomorphism the cl-group consisting of all ideals corresponds to the principal lattice ideals. By using THEOREM 2 we obtain:

$$M \sim \mathfrak{p}_1^{\alpha_1} \cdots \mathfrak{p}_n^{\alpha_n} \cdot \left(\sum_{\varkappa}' \mathfrak{q}_{\varkappa}^{-\beta_{\varkappa}}\right) \cdot \mathfrak{o}_P$$

where $P_+(J(M)) = \{\mathfrak{p}_i, \ldots, \mathfrak{p}_n\}$, $\alpha_i = \nu_{\mathfrak{p}_i}$, $P_-(J(M)) = \{\mathfrak{q}_{\varkappa}\}$, $\beta_{\varkappa} = -\nu_{\mathfrak{q}_{\varkappa}}$, P is the complement of $P_{-\infty}(J(M))$ in the set of all prime ideals not quasi-equal to $\mathfrak{o}$, $\sum'$ denotes the restricted sum, and $\mathfrak{o}_P$ is the P-component of $\mathfrak{o}$. Moreover the above decomposition is unique within quasi-equality. If in particular $\mathfrak{o}$ is Asano, each (non-zero) submodule of R is uniquely decomposed (within commutativity) as follows:

$$M = \mathfrak{p}_1^{\alpha_1} \cdots \mathfrak{p}_n^{\alpha_n} \left(\sum_{\varkappa}' \mathfrak{q}_{\varkappa}^{-\beta_{\varkappa}}\right) \mathfrak{o}_P.$$

Furthermore the P_1-component M_{P_1} of M is represented as follows:

$$M_{P_1} = \mathfrak{p}_1^{\alpha_1} \cdots \mathfrak{p}_r^{\alpha_r} \left(\sum_{\lambda}' \mathfrak{q}_{\lambda}^{-\beta_{\lambda}}\right) \mathfrak{o}_{P \vee P_1}$$

where $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} - P_1$ and $\{\mathfrak{q}_{\lambda}\} = \{\mathfrak{q}_{\varkappa}\} - P_1$ (Cf. [4] and [5].)

2. Let $\mathfrak{o}$ be a Dedekind domain withits quotient field K. Then any non-zero $\mathfrak{o}$-submodule M of K can be decomposed as in the case of the former paragraph. By using the decomposition we can prove the following statements.

The map $\varphi : x \mapsto \varphi(x)$ from a non-zero $\mathfrak{o}$-submodule M_1 to a non-zero $\mathfrak{o}$-submodule M_2 is an $\mathfrak{o}$-isomorphism if and only if there exists a non-zero element t of K such that $\varphi(x) = tx$ for all $x \in M_1$. Two non-zero $\mathfrak{o}$-submodules M_1 and M_2 are said to have the same $-\infty$-type iff $\mathfrak{o}_{P_{-\infty}(M_1)} = \mathfrak{o}_{P_{-\infty}(M_2)}$. Then in order that M_1 and M_2 have the same $-\infty$-type, it is necessary and sufficient that there is an ideal $\mathfrak{a}$ such that $M_2 = M_1\mathfrak{a}$. Let $\mathfrak{m}$ be the ideal generated by all prime ideals in $P_{-\infty}(M)$, and let $\mathfrak{a}$ be an ideal. Then M is $\mathfrak{o}$-isomorphic to $M\mathfrak{a}$ if and only if $\mathfrak{a}$ is represented as $\mathfrak{a} = \mathfrak{m}(a)$ for a non-zero element a of K. Any intermediate ring T of $\mathfrak{o}$ and K is a P-component of $\mathfrak{o}$, and it is a Dedekind ring. An integral T-ideal $\mathfrak{P}$ of T is prime if and only if $\mathfrak{P} = \mathfrak{p}T$, where $\mathfrak{p}$ is a prime ideal in $P_0(T)$.

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